SMOOTH MANIFOLDS FALL 2023 - HOMEWORK 4.5

Problem 1. Consider spheres with centers $x_1, x_2 \in \mathbb{R}^3$ and radii $r_1, r_2 \in \mathbb{R}_+$, respectively. Find the quadruples (x_1, x_2, r_1, r_2) for which the intersection is transverse, and show that your answer is correct.

Problem 2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^∞ function. Find a diffeomorphism between $\mathbb{R}^2$ and Γ_f, the graph of f. Compute the tangent bundle to the graph of f in two ways:

(1) at each point $z \in \Gamma_f$, find a pair of linearly independent vectors in $T_z\Gamma_f$
(2) at each point $z \in \Gamma_f$, find a functional φ_z such that $T_z\Gamma_f = \ker \varphi_z$

Finally, consider two C^∞ functions f and g on $\mathbb{R}^2$. Find a necessary and sufficient condition on the functions f and g so that there exists $c \in \mathbb{R}$ such that Γ_f and Γ_{g+c} intersect nontrivially and transversally.

Problem 3. Show that for almost every matrix $A \in M_2(\mathbb{R}) = \mathbb{R}^4$, the function $F_A : S^1 \to \mathbb{R}^2$ is transverse to S^1, where $F_A(v) = Av$. For which matrices A does the image of F_A intersect S^1 nontrivially? For which values is F_A an immersion? An embedding?

Problem 4. Let $X = M_2(\mathbb{R}) = \mathbb{R}^2$, and $F : X \to X$ denote the function $F(A) = A^2$. Compute the differential of the function F as a 4×4 matrix.